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The Length of the Alternation Set as a Factor in Determining When a Best Real Rational Approximation Is Also a Best Complex Rational Approximation

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The purpose of this note is to characterize when the length of the alternation set can be used to determine if a best real rational approximant of a real continuous function on a real interval is also a best complex rational approximant.

1. INTRODUCTION

For a given pair (m, n) of non-negative integers and a given continuous real function f on $[-1, 1]$, it is well known that there is a unique best uniform approximation from the set of real rational functions with numerator degree at most m and denominator degree at most n and that the best approximation R is characterized by the length of the alternation set (defined below) of $f-R$. See, for instance, [2, p. 161]. Less is known about complex rational approximants of real functions. One might expect as in the polynomials case that admitting complex approximants would not produce a better approximation since the imaginary part of the rational function does not aid in approximating a real function. It seems to be a fairly recent observation that that is not the case. See, for example, [4], which gives as a special case the earlier result of [1].

This prompts one to wonder under what circumstance a best real rational approximant is also a best complex approximant. In [4], Saff and Varga gave a partial answer to that question. Their result, stated as Theorem 1.1, below, gives two constants d_1 and d_2 such that if the length of very alternation set of $f-R$ is no more than d_1 then R is not the best complex rational approximant and if $f-R$ has a alternation set of length at least d_2 then R is a best complex rational approximant. Unfortunately, in most cases $d_1 < d_2$ so that their results left a gap in which one could not decide on the basis of

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the alternation set alone whether or not R was a best complex rational approximant. Later, Wulbert observed in [5] that this gap cannot be removed. That is, in certain circumstances a knowledge of the length of the alternation set of $f-R$ is not sufficient to decide if R is a best complex rational approximant of f.

The main result of this note, Theorem 3.4, explicitly describes that gap. We prove Theorem 3.4 by first showing that d_1 can be replaced by $m+n+1$ (Theorem 2.1) and then proving that the constants $m+n+1$ and d_2 cannot be improved (Corollary 2.4 and Theorem 3.3).

To give a precise statement of the result of Saff and Varga and of our extension of that result, we will need to develop some notation. For any non-negative integer n, let Π_n denote the set of all polynomials with real or complex coefficients which have degree at most n, and let Π_n^r be the subset of Π_n which consists of polynomials with only real coefficients. We will use $\Pi_{m,n}$ and $\Pi_{m,n}^r$ to represent the sets $\{q/p\colon q \in \Pi_m \text{ and } p \in \Pi_n\}$ and $\{q/p\colon q \in \Pi_m^r \text{ and } p \in \Pi_n^r\}$, respectively. For arbitrary polynomials q and p, let (q,p) be the greatest common divisor of q and p. In particular, if $(q,p)=1$ then p and q have no common factors.

For a given real or complex function f, let $\|f\| := \sup_{x\in[-1,1]} |f(x)|$. A rational function $R \in \Pi_{m,n}(\Pi_{m,n}^r)$ is a best uniform approximation of f from $\Pi_{m,n}(\Pi_{m,n}^r)$ if $\|f-R\| = \inf_{T\in\Pi_{m,n}} \|f-T\|$ $(\inf_{T\in\Pi_{m,n}^r} \|f-T\|)$. The collection of all best uniform approximations of f from $\Pi_{m,n}(\Pi_{m,n}^r)$ will be denoted by $B_{m,n}(f)(B_{m,n}^r(f))$.

If f is a real continuous function on $[-1,1]$, known compactness arguments show that $B_{m,n}(f)$ and $B_{m,n}^r(f)$ are not empty. In fact, as we mentioned from above, $B_{m,n}^r(f)$ contains a single element $R=q/p$ with $(q,p)=1$ which is characterized by the property that $f-R$ has an alternation set containing at least $2+\max(m+\deg p, n+\deg q)$ points, where for any real function g on $[-1,1]$ an alternation set for g is defined to be any finite collection $x_1 < x_2 < \cdots < x_k$ of points of $[-1,1]$ such that $x_j \in \operatorname{crit}(g) := \{x \in [-1,1]\colon |g(x)| = \|g\|\}$, $j=1,2,\ldots,k$ and

$$g(x_j)\, g(x_{j+1}) \leqslant 0, \qquad j=1,2,\ldots,k-1.$$

With this definition, the result of Saff and Varga can be stated as follows:

THEOREM 1.1. *Let f be a real continuous function on $[-1,1]$ and let $R \in B_{m,n}^r(f)$, where $R=q/p$ and $(q,p)=1$.*

(a) *If every alternation set of $f-R$ contains at most $d_1 := 1+m+\min(n-\deg p, m-\deg q)$ elements, then $R \notin B_{m,n}(f)$.*

(b) *If $f-R$ has an alternation set which contains at least $d_2 := 2+n+\max(m+\deg p, n+\deg q)$ points, then $R \in B_{m,n}(f)$.*

In proving Theorem 1.1, Saff and Varga obtained a slightly stronger result than is stated in Theorem 1.1(a). They showed that if every alternation set of $f-R$ contains at most d_1 elements, then for any $\varepsilon > 0$ there is an $R_\varepsilon \in \Pi_{m,n}$ for which $\|f-R_\varepsilon\| < \|f-R\|$ and $\|R-R_\varepsilon\| < \varepsilon$. That is, R is not even a local best approximation of f. Defining $B^{\ell}_{m,n}(f)$ to be the set of $T \in \Pi_{m,n}$ for which there is an $\varepsilon > 0$ (which depends on T) such that if $S \in \Pi_{m,n}$ and if $\|f-S\| < \|f-T\|$ then $\|S-T\| > \varepsilon$, we see that Saff and Varga really proved that $R \notin B^{\ell}_{m,n}(f)$. Clearly $B_{m,n}(f) \subseteq B^{\ell}_{m,n}(f)$. We shall state our extension of Theorem 1.1(a) in terms of $B^{\ell}_{m,n}(f)$.

2. Properties of $B^{\ell}_{m,n}(f)$ and $B_{m,n}(f)$

In this section, we will establish two necessary conditions for a rational function R to be a member of $B^{\ell}_{m,n}(f)$, Theorem 2.1 and Theorem 2.2; and one sufficient condition for R to be a member of $B_{m,n}(f)$, Theorem 2.3. As we remarked above, Theorem 2.1 shows that d_1 of Theorem 1.1(a) can be replaced by $m+n+1$.

THEOREM 2.1. *Let f be a continuous real function on $[-1,1]$, and let $R \in B^{r}_{m,n}(f)$, where $R = q/p$ and $(q,p) = 1$. If every alternation set of $e := f - R$ contains at most $m+n+1$ elements, then $R \notin B^{\ell}_{m,n}(f)$.*

Proof. We shall consider rational functions of the form

$$R_\lambda := \frac{sq + \lambda\alpha}{sp + \lambda\beta}$$

where $\deg s \leqslant d := \min\{n - \deg p, m - \deg q\}$, $\alpha \in \Pi_m$, $\beta \in \Pi_n$, and λ is a small positive real number. With s, α, and β satisfying the conditions above, it is easily verified that $R_\lambda \in \Pi_{m,n}$. Our goal is to show that we may choose s, α, and β so that

$$e \operatorname{Re}\{\bar{s}(\beta q - \alpha p)\} < 0 \qquad \text{on} \quad \operatorname{crit}(e), \tag{2.1.1}$$

where $\bar{s}$ is the complex conjugate of s, and

$$s \text{ has no zeros in } [-1, 1]. \tag{2.1.2}$$

Given s, α, and β satisfying (2.1.1) and (2.1.2), we will have that

$$\|f - R_\lambda\| < \|f - R\| \tag{2.1.3}$$

for all λ sufficiently small, whence $R \notin B^{\ell}_{m,n}(f)$.

To see that (.2.1.1) and (2.1.2) imply (2.1.3), we first note that since $R \in B^r_{m,n}(f)$ we may assume $\min_{x \in [-1,1]} p(x) > 0$. That assumption, (2.1.1), the compactness of crit(e), and continuity guarantee that there is an open set $\mathscr{U}$ of $[-1, 1]$ containing crit(e) such that for some $\varepsilon > 0$

$$e(x) \operatorname{Re}\{\overline{s(x)}(\beta(x)\, q(x) - \alpha(x)\, p(x))\} < -\varepsilon \qquad \text{for } x \in \mathscr{U} \tag{2.1.4}$$

In addition, that assumption and (2.1.2) yields that

$$0 < \mu := \tfrac{1}{2} \min_{x \in [-1,1]} |s(x)\, p(x)|^2 < \min_{x \in [-1,1]} |s(x)\, p(x) + \lambda\beta(x)|^2, \tag{2.1.5}$$

and

$$0 < \tfrac{1}{2} \|sp\|^2 \leqslant \|sp + \lambda\beta\|^2 < 2\, \|sp\|^2 := \gamma \tag{2.1.6}$$

for all λ sufficiently small, say $0 < \lambda < \lambda_0$. Defining $t := \beta q - \alpha p$, we obtain

$$\begin{aligned} &|f(x) - R_\lambda(x)|^2 - |f(x) - R(x)|^2 \\ &= \frac{\lambda}{|s(x)\, p(x) + \lambda\beta(x)|^2} \left[2e(x) \operatorname{Re}\{\overline{s(x)}\, t(x)\} \right. \\ &\quad \left. + 2\lambda e(x) \operatorname{Re}\left\{ \frac{t(x)\, \overline{\beta(x)}}{p(x)} \right\} + \lambda \frac{|t(x)|^2}{p(x)^2} \right]. \end{aligned} \tag{2.1.7}$$

Let $M := \|2e \operatorname{Re}(t\bar{\beta}/p) + (|t|^2/p^2)\|$, and assume $\lambda < \lambda_0$. From Eq. (2.1.6) and inequality (2.1.4), we conclude that

$$|f(x) - R_\lambda(x)|^2 \leqslant \|f - R\|^2 - 2\frac{\lambda}{\gamma}\varepsilon + \frac{\lambda^2 M}{\mu} \qquad \text{for } x \in \mathscr{U}. \tag{2.1.8}$$

If $\lambda < \min\{\lambda_0, \mu\varepsilon/M\gamma\}$, then (2.1.8) implies

$$|f(x)| - R_\lambda(x)|^2 \leqslant \|f - R\|^2 - \frac{\lambda}{\gamma}\varepsilon \qquad \text{for } x \in \mathscr{U}. \tag{2.1.9}$$

On the other hand, since crit$(e) \subset \mathscr{U}$, it follows that there is a $\delta > 0$ such that

$$|f(x) - R(x)|^2 \leqslant \|f - R\|^2 - 2\delta \qquad \text{for all} \quad x \in [-1, 1] - \mathscr{U}. \tag{2.1.10}$$

If we let $K := \|2e \operatorname{Re}\{\bar{s}t\}\|$, then (2.1.7) and (2.1.10) give

$$|f(x) - R_\lambda(x)|^2 \leqslant \|f - R\|^2 - 2\delta + \frac{\lambda K}{\mu} + \frac{\lambda^2 M}{\mu} \qquad \text{for} \quad x \in [-1, 1] - \mathscr{U}. \tag{2.1.11}$$

By choosing $\lambda < \min\{\lambda_0, \mu\varepsilon/M, \delta\mu\gamma/(K\gamma + \mu\varepsilon)\}$ and using (2.1.11) we find that

$$|f(x) - R_\lambda(x)|^2 \leqslant \|f - R\|^2 - \delta \qquad \text{for all} \quad x \in [-1, 1] - \mathscr{U}. \tag{2.1.12}$$

Hence by (2.1.9) and (2.1.12), inequality (2.1.3) is established.

The proof will be complete if we show that s, α, and β can be chosen so that (2.1.1) and (2.1.2) hold. Let k be the length of the longest alternation set of e. Since $k \leqslant m + n + 1$, the best approximation w of e from Π^r_{m+n} on $[-1, 1]$ is not identically zero. So, for $x \in \text{crit}(e)$,

$$(e(x) - w(x))^2 = e(x)^2 - 2e(x)\,w(x) + w(x)^2 < e(x)^2,$$

and therefore we have that $e(x)\,w(x) > 0$ for $x \in \text{crit}(e)$. Consider a polynomial of the form $v(x) = -w(x + \delta i)$, where $\delta > 0$ is small. It follows that for δ sufficiently small

$$e(x)\,\text{Re}\, v(x) < 0 \quad \text{for all} \quad x \in \text{crit}(e), \tag{2.1.13}$$

and

$$v(x) \text{ has no zeros in } [-1, 1]. \tag{2.1.14}$$

Take δ small enough so that (2.1.13) and (2.1.14) hold. Since $v \in \Pi_{m+n}$, v may be factored into the product of polynomials $\hat{s}$ and $\hat{t}$, where $\deg \hat{s} \leqslant d := \min\{n - \deg p, m - \deg q\}$ and $\deg \hat{t} \leqslant N := \max\{m + \deg p, n + \deg q\}$. Say $\hat{s}(x) = \sum_{j=0}^d a_j x^j$. Define $s(x) := \sum_{j=0}^d \bar{a}_j x^j$. Since v is not zero in $[-1, 1]$, neither are $\hat{s}$ and s. Further, since $\hat{t} \in \Pi_N$ and p and q are relatively prime, there are polynomials $\alpha \in \Pi_m$, $\beta \in \Pi_n$ for which $\hat{t} = \beta q - \alpha p$. But on crit$(e)$, $\overline{s(x)}(\beta(x)\,q(x) - \alpha(x)\,p(x)) = \hat{s}(x)\,\hat{t}(x) = v(x)$, and therefore (2.1.1) and (2.1.2) follow from (2.1.13) and (2.1.14). ∎

If $R \in B^r_{m,n}(f)$ then R is the best approximation of f on crit$(f - R)$ from the subset of functions from $\Pi^r_{m,n}$ which are also continuous on $[-1, 1]$ (cf. Rivlin [3, p. 131]). However, R need not be the best approximation of f on crit$(f - R)$ from $\Pi^r_{m,n}$. For example, let $f(x) = x - 2T_3(x)$ where T_3 is the Chebyshev polynomial of degree 3, that is, $f(x) = 7x - 8x^3$. Note that $f(x) - x = -2T_3(x)$ has an alternation set of length 4, and therefore $x \in B^r_{1,1}(f)$. But, an easy calculation shows that on crit$(f - x) = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$, $|f(x) - x| = 2$ while $|f(x) - 1/2x| = \frac{3}{2}$. Combining these two functions to form

$$R_\lambda = \frac{x + \lambda i}{1 + 2x\lambda i}, \qquad \lambda > 0,$$

a direct calculation shows that $\|f-R_\lambda\| < \|f-x\|$ for λ sufficiently small. That fact motivates the following theorem:

THEOREM 2.2. *Let f be a continuous real function on $[-1, 1]$. Let $R \in B^r_{m,n}(f)$, $R = q/p$ where $(p, q) = 1$, and $e := f - R$. If there exists $T \in \Pi^r_{m,n}$ such that*

$$|f(x) - T(x)| < \|e\|, \qquad x \in \operatorname{crit}(e) \tag{2.2.1}$$

then $R \notin B^{\ell}_{m,n}(f)$.

Proof. Suppose (2.2.1) holds. Say $T = \alpha/\beta$, where $\alpha \in \Pi^r_m$, $\beta \in \Pi^r_n$, and $(\alpha, \beta) = 1$. For $x \in \operatorname{crit}(e)$,

$$\begin{aligned} 0 &> |f(x) - T(x)|^2 - |e(x)|^2 \\ &= 2e(x) \left[\frac{\beta(x)\, q(x) - \alpha(x)\, p(x)}{p(x)\, \beta(x)}\right] + \left[\frac{\beta(x)\, (x) - \alpha(x)\, p(x)}{p(x)\, \beta(x)}\right]^2 \\ &= \frac{1}{\beta(x)^2} \left[2e(x) \frac{t(x)\, \beta(x)}{p(x)} + \frac{t(x)^2}{p(x)^2}\right], \end{aligned}$$

where $t := \beta q - \alpha p$. So

$$2e(x) \frac{t(x)\, \beta(x)}{p(x)} + \frac{t(x)^2}{p(x))^2} < 0 \qquad \text{for} \quad x \in \operatorname{crit}(e). \tag{2.2.2}$$

For real λ, we define

$$R_\lambda := \frac{q + \lambda i \alpha}{p + \lambda i \beta}.$$

Since $\deg(q + \lambda i \alpha) \leqslant m$ and $\deg(p + \lambda i \beta) \leqslant n$, for all λ, $R_\lambda \in \Pi_{m,n}$. Moreover, since $R \in B^r_{m,n}(f)$, we may assume that

$$0 < \mu := \min_{x \in [-1,1]} p(x), \tag{2.2.3}$$

and therefore there is a $\lambda_0 > 0$ such that

$$\mu \leqslant \min_{x \in [-1,1]} |p(x) + i\lambda\beta(x)| < 2\mu \qquad \text{for all} \quad |\lambda| < \lambda_0. \tag{2.2.4}$$

Now,

$$|f - R_\lambda|^2 - |f - R|^2 = \frac{\lambda^2}{|p + i\lambda\beta|^2} \left\{ 2e \frac{t\beta}{p} + \frac{t^2}{p^2} \right\}. \tag{2.2.5}$$

By (2.2.2), (2.2.3), and (2.2.4), the right side of (2.2.5) is continuous on $[-1, 1]$ and negative on the compact set crit(e). Thus, there is an open set $\mathscr{U}$ of $[-1, 1]$ containing crit(e) and an $\varepsilon > 0$ such that

$$|f(x) - R_\lambda(x)|^2 < |f(x) - R(x)|^2 - \frac{\lambda^2}{2\mu}\varepsilon \qquad \text{for} \quad |\lambda| < \lambda_0, \quad x \in \mathscr{U}. \tag{2.2.6}$$

Proceeding as in Theorem 2.1, we find a $\delta > 0$ and a $\lambda_1 > 0$ for which

$$|f(x) - R_\lambda(x)|^2 < \|f(x) - R(x)\|^2 - \delta \qquad \text{for} \quad |\lambda| < \lambda_1, \quad x \in [-1, 1] - \mathscr{U}. \tag{2.2.7}$$

From inequalitites (2.2.7) and (2.2.6), we obtain

$$\|f - R_\lambda\| < \|f - R\|$$

for all λ sufficiently small which establishes the result. ∎

We have claimed that the constant $m + n + 1$ of Theorem 2.1 cannot be replaced by a larger constant and still have the conclusion of the theorem by valid. That fact is a consequence of the next two results, Theorem 2.3 and Corollary 2.4. Actually, Theorem 2.3 is an extension of a result due to Wulbert. In [5], he proved Theorem 2.3 for the special case when R is normal ($\deg q = m$ or $\deg p = n$).

THEOREM 2.3. *Let f and R be continuous real-valued functions on $[-1, 1]$ such that $R \in \Pi^r_{m,n}$, $R = q/p$, $(q, p) = 1$, and $e := f - R$ has an alternation set of length at least $m + n + 2$. Then*

$$R \in B_{m,n}(\lambda e + R) \qquad \textit{for all sufficiently small } \lambda > 0. \tag{2.3.1}$$

Proof. If $q \equiv 0$, we conclude from Theorem 1.1 that (2.3.1) holds. Assume that $q \not\equiv 0$ and that (2.3.1) does not hold. With that assumption, we must have a sequence $\{\lambda_k\}$ with $\lambda_k > 0$, $\lambda_k \to 0$ as $k \to \infty$ and a sequence $\{T_k\}$ with $T_k \in \Pi_{m,n}$, $T_k = s_k/t_k$, $(s_k, t_k) = 1$ such that

$$\|\lambda_k f + (1 - \lambda_k) R - T_k\| < \lambda_k \|f - R\|. \tag{2.3.2}$$

We begin by writing T_k in a more convenient form. Suppose that for some fixed k, $\deg t_k - \deg p \leqslant \deg s_k - \deg q$. Using the division algorithm for polynomials, we find polynomials u_k, β_k such that

$$t_k := u_k p + \beta_k \qquad \text{with} \qquad \deg \beta_k < \deg p.$$

Defining $\alpha_k := s_k - u_k q$, we see that $\alpha_k \in \Pi_m$ and

$$T_k = \frac{u_k q + \alpha_k}{u_k p + \beta_k}, \tag{2.3.3}$$

where

$$\alpha_k \in \Pi_m, \qquad \beta_k \in \Pi_n$$

and (2.3.4)

$$\deg \alpha_k < \deg q \qquad \text{or} \qquad \deg \beta_k < \deg p.$$

If $\deg t_k - \deg p > \deg s_k - \deg q$, we apply the division algorithm to s_k and q to obtain $s_k = u_k q + \alpha_k$. By setting $\beta_k := t_k - u_k p$, we obtain (2.3.3), and (2.3.4). Note that $\deg u_k \leqslant \min\{m - \deg q, n - \deg p\}$.

Let $\varepsilon_k := \max\{\|\alpha_k\|, \|\beta_k\|\}$. Define $\delta_k := \alpha_k/\varepsilon_k$, $\gamma_k := \beta_k/\varepsilon_k$, and $r_k := \gamma_k q - \delta_k p$. With those definitions, we have

$$\|\delta_k\| = 1 \qquad \text{or} \qquad \|\gamma_k\| = 1, \tag{2.3.5}$$

and

$$T_k = \frac{u_k q + \varepsilon_k d_k}{u_k p + \varepsilon_k \gamma_k}. \tag{2.3.6}$$

In addition, since $\{\|\gamma_k\|\}$ and $\{\|\delta_k\|\}$ are bounded sequences, by passing to an appropriate subsequence we may assume that $\{\delta_k\}$ and $\{\gamma_k\}$ are both convergent. Say $\delta_k \to \delta$ and $\gamma_k \to \gamma$. By (2.3.4) and (2.3.5), we may also assume that either $\deg \delta < \deg q$ or $\deg \gamma < \operatorname{dep} p$ and either $\|\delta\| = 1$ or $\|\gamma\| = 1$.

Let $r := \lim_{k \to \infty} r_k = \gamma q - \delta p$. If $r \equiv 0$ then since $(q, p) = 1$, $q \not\equiv 0$, and $p \not\equiv 0$, it must follow that p divides γ and q divides δ. But that cannot happen since $\deg \gamma < \deg p$ or $\deg \delta < \deg q$ and at least one of γ and δ is not identically zero. We conclude that $r \not\equiv 0$. In particular, since crit(e) contains at least $m + n + 2$ points and $r \in \Pi_N$, $N := \max(m + \deg p, n + \deg q)$, we observe that

$$\max_{x \in \operatorname{crit}(e)} |r(x)| > 0. \tag{2.3.7}$$

We will now obtain a contradiction from (2.3.2). By (2.3.2) and (2.3.6), we have that for $x \in \operatorname{crit}(e)$

$$\begin{aligned}0 &> |\lambda_k e(x) + R(x) - T_k(x)|^2 - \lambda_k^2 |e(x)|^2 \\ &= 2\lambda_k e(x) \operatorname{Re}\{R(x) - T_k(x)\} + |R(x) - T_k(x)|^2 \\ &= \frac{1}{|t_k(x)|^2} \left[2\lambda_k \varepsilon_k e(x) \operatorname{Re}\{\overline{u_k(x)}\, r_k(x)\} \right. \\ &\quad \left. + 2\lambda_k \varepsilon_k^2 e(x) \operatorname{Re}\left\{ \frac{r_k(x)\,\overline{\gamma_k(x)}}{p(x)} \right\} + \frac{\varepsilon_k^2 |r_k(x)|^2}{p(x)^2} \right].\end{aligned}$$

So for $x \in \operatorname{crit}(e)$,

$$\begin{aligned}0 &> 2e(x) \operatorname{Re}\left\{ \frac{\lambda_k u_k(x)\, r_k(x)}{\varepsilon_k} \right\} \\ &\quad + 2\lambda_k e(x) \operatorname{Re}\left\{ \frac{r_k(x)\,\overline{\gamma_k(x)}}{p(x)} \right\} + \frac{|r_k(x)|^2}{p(x)^2}.\end{aligned} \tag{2.3.8}$$

For x real, $w_k := \operatorname{Re}(\lambda_k \bar{u}_k r_k/\varepsilon_k)$ is a real polynomial with degree at most $m+n$, and consequently (2.3.8) becomes

$$0 > 2e(x)\, w_k(x) + 2\lambda_k e(x) \operatorname{Re}\left\{ \frac{r_k(x)\,\overline{\gamma_k(x)}}{p(x)} \right\} + \frac{|r_k(x)|^2}{p(x)^2}, \qquad x \in \operatorname{crit}(e) \tag{2.3.9}$$

where $w_k(x) \in \Pi^r_{m+n}$ and $r_k \in \Pi_N$.

If $\overline{\lim}_{k\to\infty} \|w_k\| = 0$, then letting $k \to \infty$ in (2.3.9) we obtain

$$0 \geqslant \frac{|r(x)|^2}{p(x)^2} \qquad \text{for all } x \in \operatorname{crit}(e)$$

contradicting (2.3.7). If $\overline{\lim}_{k\to\infty} \|w_k\| \neq 0$, by choosing a subsequence if necessary we may assume that $w_k/\|w_k\|$ converges to a polynomial $w \in \Pi^r_{m+n}$ and that $\lim_{k\to\infty} 1/\|w_k\|$ exists and is finite. Dividing (2.3.9) by $\|w_k\|$ and passing to the limit, we find that

$$0 \geqslant e(x)\, w(x), \qquad x \in \operatorname{crit}(e).$$

But $\operatorname{crit}(e)$ contains an alternation set for e with length at least $m+n+2$. By the standard argument, we conclude that w has at least $m+n+1$ zeros. As $w \in \Pi_{m+n}$, w must be identically zero. However, $\|w\| = 1$, and therefore, we have obtained a contradiction. Hence (2.3.1) is established. ∎

COROLLARY 2.4. *For any integers m, n, d_1, d_2, k* with $0 \leqslant d_1 \leqslant n$, $0 \leqslant d_2 \leqslant m$, *and* $k \geqslant m+n+2$, *there is a real continuous function f on*

$[-1, 1]$ *and a real continuous rational function* $R \in \Pi^r_{m,n}$ *for which* $R = q/p$, $(q, p) = 1$, *degree* $p = d_1$, *degree* $q = d_2$, *the longest alternation set of* $f - R$ *has length* k, *and* $R \in B_{m,n}(f)$.

Proof. Let e be a real continuous function on $[-1, 1]$ whose longest alternation set has length k. Choose $q \in \Pi^r_m$ and $p \in \Pi^r_n$ so that p and q have degree d_1 and d_2, respectively, $(p, q) = 1$, and p has no zeros in $[-1, 1]$. Let $R := q/p$. By Theorem 2.3, for $\lambda > 0$ sufficiently small, $R \in B_{m,n}(\lambda e + R)$. Setting $f := \lambda e + R$ gives the result. ∎

3. The Main Result

In this section we first prove that the constant d_2 of Theorem 1.1(b) cannot be improved. We do this by showing in Theorem 3.2 that for any positive integer k with $k < d_2$ there is a real continuous function f and a real rational $R \in \Pi^r_{m,n}$ such that $R \in B^r_{m,n}(f)$ and the longest alternation set of $f - R$ has length k but $R \notin B^l_{m,n}(f)$. Our main result, Theorem 3.4, will then follow as consequence of Theorem 1.1, Theorem 1.2, and Theorem 3.2.

Lemma 3.1. *Let* k, ℓ *be integers with* $k \geqslant \ell \geqslant 0$. *There exist real polynomials* p_1 *and* p_2 *such that*

(a) $\deg p_1 = k$, $\deg p_2 = \ell$, *and* $(p_1, p_2) = 1$;

(b) p_1 *and* p_2 *have respectively* k *and* ℓ *distinct zeros in* $(-1, 1)$;

(c) $\deg(p_1 + p_2) = k$; *and*

(d) $p_1 + p_2$ *has no zeros in* $[-1, 1]$ *and* $p_1 + p_2$ *has at least one real zero when* $k \neq 0$.

Proof. When $k = 0$, put $p_1 = p_2 = 1$. If $k \neq 0$ we consider three cases: k odd, k even and ℓ odd, k even and ℓ even.

Case 1. k odd.

Let $p_1 = T_k(2x + 1)$ and $p_2 = (-1)^\ell c T_\ell(2x - 1)$, where T_k and T_ℓ are Chebyshev polynomials of degree k and ℓ, respectiverly, and c is an arbitrary constant, $0 < c < 1$. Since p_1 has k zeros in $(-1, 0)$ and p_2 has ℓ zeros in $(0, 1)$, it is evident that (a)–(c) are satisfied. We need to determine c so that (d) holds.

Recall that the Chebyshev polynomial of degree j, T_j, satisfies $|T_j(x)| \leqslant 1$ when $x \in [-1, 1]$, $|T_j(x)| > 1$ when $x \notin [-1, 1]$ and $T_j(-1) = (-1)^j$. When $c = 1$ that fact gives

$$|p_1(x)+p_2(x)| \geqslant |T_\ell(2x-1)| - |T_k(2x+1)| > 0$$
$$\text{for } x \in [-1, 0),$$
$$|p_1(x)+p_2(x)| \geqslant |T_k(2x+1)| - |T_\ell(2x-1)| > 0$$
$$\text{for } x \in (0, 1],$$

and

$$p_1(0)+p_2(0) = 2.$$

But when $c = 1$, (c) is not satisfied for $\ell = k$. By taking $c = 1-\varepsilon$, $\varepsilon > 0$ sufficiently small, $p_1 + p_2$ will still have no zeros in $[-1, 1]$ and $\deg(p_1+p_2) = k$ for any ℓ, $k \geqslant \ell \geqslant 0$. Since p_1+p_2 has odd degree, p_1+p_2 must have at least one real zero.

Case 2. k even and ℓ odd.

Let $q_1 := (x+\frac{1}{2})^k$, $q_2 := -c(x-1)^\ell$, where $c > 1$. Clearly, $q+q_2$ has no zeros in $[-1, 1]$. Let $c = (\frac{5}{2})^k$ so that q_1+q_2 has a simple zero at 2. By continuity, we may choose distinct points $-1 < y_1 < y_2 < \cdots < y_k < z_1 < z_2 < \cdots < z_\ell < 1$ so that if $|y_j + \frac{1}{2}|$, $j = 1, 2, ..., k$, and $|z_j - 1|$, $j = 1, 2, ..., \ell$ are sufficiently small then

$$s(x) := \prod_{j=1}^{k} (x-y_j) - (\tfrac{5}{2})^k \prod_{j=1}^{\ell} (x-z_j)$$

has no zeros in $[-1, 1]$, and s has a zero in $(1, \infty)$. Setting $p_1 = \prod_{j=1}^k (x-y_j)$ and $p_2 = -(\frac{5}{2})^k \prod_{j=1}^{\ell} (x-z_j)$ establishes (a)–(d).

Case 3. k even and ℓ even.

Choose positive integers s and t such that s and t are odd and $s+t=k$. Define $q_1(x) := (x-1)^s\,(x+1)^t$, $q_2(x) := -\frac{1}{2}x^\ell$. Then q_1+q_2 has no zeros in $[-1, 1]$. Moreover, since $q_1(1)+q_2(1) = -\frac{1}{2}$ and $q_1(x)+q_2(x) > 0$ for all x sufficiently large it follows that $q_1(x)+q_2(x)$ has an odd zero in $(1, \infty)$. Proceeding as in Case 2, we obtain (a)–(d). ∎

LEMMA 3.2. *For any integers m, n, d_1, d_2 with $0 \leqslant d_1 \leqslant n$, $0 \leqslant d_2 \leqslant m$, there are real rational functions R and T satisfying*

(a) *$R = q/p$, where* $\deg q = d_2$, $\deg p = d_1$, $(q, p) = 1$, *and* $p(x) \neq 0$ *for* $x \in [-1, 1]$;

(b) *$T = s/t$, where* $\deg s \leqslant m$, $\deg t \leqslant n$, $(s, t) = 1$;

(c) *there are points $-1 < x_1 < x_2 < \cdots < x_L < 1$, where $L := 1 + n + \max\{n+d_2, m+d_1\}$ such that*

$$[R(x_j) - T(x_j)][R(x_{j+1}) - T(x_{j+1})] < 0, \qquad j = 1, 2, ..., L-1$$

and $t(x_j) \neq 0$, $j = 1, 2, ..., L$.

Proof. First suppose $n+d_2 \leqslant m+d_1$. Define $N := \max\{n+d_2, m+d_1\}$, $k := N-d_2$, and $\ell := n$. Let p_1 and p_2 be polynomials satisfying (a)–(d) of Lemma 3.1. Choose any polynomial $q \in \Pi^r_{d_2}$ which has d_2 distinct zeros in $(-1, 1)$ different than those of p_1 and p_2. Denote $-p_2$ by t. Condition (d) of Lemma 3.1 guanrantees that $-p_1+t$ can be factored into the product of real polynomials r and p where $\deg r = k-d_1$ and $\deg p = d_1$ and p has no zeros in $[-1, 1]$. Putting $s := qr$, we observe that $\deg s \leqslant d_2+k-d_1 = m$.

Now, define $R := q/p$ and $T := s/t$. Then

$$R - T = \frac{qt - sp}{pt} = \frac{qp_1}{pt},$$

which has exactly N distinct simple zeros and exactly n distinct simple poles in $(-1, 1)$. Let $y_1 < y_2 < \cdots < y_{L-1}$ represent all of those zeros and poles, let $y_0 := -1$, and let $y_L = 1$. Since the zeros and poles of $R-T$ in $[-1, 1]$ are simple, $R-T$ does not change sign on (y_j, y_{j+1}), $j = 0, 1, \ldots, L-1$, and moreover, $R-T$ has different signs on (y_j, y_{j+1}) and (y_{j+1}, y_{j+2}), $j = 1, \ldots, L-2$. Selecting $x_j \in (y_{j-1}, y_j)$, $j = 1, 2, \ldots, L$ yields (a)–(c).

If $N := n+d_2 > m+d_1$, let $p_1 \in \Pi^r_N$ be any real polynomial with N distinct zeros in $(-1, 1)$, let p be any polynomial with $\deg p = d_1$ such that $p(x) > 0$ for all $x \in [-1, 1]$, and let $s > 0$ be a constant so small that $p_1 + sp$ has N distinct zeros in $(-1, 1)$. Our choice of p_1 and p implies that p_1 and $p_1 + sp$ have no common zeros. Factor the polynomial $p_1 + sp$ into the product of polynomials q and t so that $\deg t = n$ and $\deg q = d_2$. Define $R := q/p$ and $T := s/t$. Since $R - T = p_1/pt$, $R-T$ has $N+n = L-1$ distinct simple zeros and poles in $(-1, 1)$. Continuing as above, we obtain (a)–(c). ∎

THEOREM 3.3. *For any integers m, n, d_1, d_2, k with $0 \leqslant d_1 \leqslant n$, $0 \leqslant d_2 \leqslant m$, and $0 < k \leqslant L := 1 + n + \max\{n+d_2, m+d_1\}$, there is a real rational function R with no poles in $[-1, 1]$ and a continuous real function f on $[-1, 1]$ for which*

(a) *$R = q/p$ where $\deg q = d_2$, $\deg p = d_1$, and $(q, p) = 1$;*

(b) *$f - R$ has an alternation set of length k and $R \notin B^{\ell}_{m,n}(f)$.*

Proof. Let R and T be as in Lemma 3.2. By (c) of Lemma 3.2 there is a set of points $X := \{x_1 < x_2 < \cdots < x_k\}$ such that

$$\begin{aligned}\operatorname{sign}[R(x_j) - T(x_j)]\\ = (-1)^{j+1} \operatorname{sign}[R(x_1) - T(x_1)], \qquad j = 1, 2, \ldots, k.\end{aligned}$$

Define $\hat{e}$ on X by

$$\hat{e}(x_j) := (-1)^j \lambda,$$

where

$$\lambda := \operatorname{sign}[R(x_1) - T(x_1)]\{1 + \max_{1 \leqslant j \leqslant k} \{R(x_j) - T(x_j)\}.$$

Let e be any continuous extension of $\hat{e}$ to $[-1, 1]$ such that

$$\|e\| \leqslant \lambda \qquad \text{and} \qquad |e(x)| < \lambda \qquad \text{for } x \neq x_1, ..., x_k.$$

Now, put $f := e + R$. Since both e and R are continuous, so is f. Also $f - R$ has an alternation set of length k. But for $x \in X$,

$$\begin{aligned} |f(x_j) - T(x_j)| &= |e(x_j) + R(x_j) - T(x_j)| \\ &= |(-1)^j \lambda + R(x_j) - T(x_j)| \qquad (3.3.1) \\ &= ||\lambda| - |R(x_j) - T(x_j)|| < |\lambda|. \end{aligned}$$

However, (3.3.1) implies that T is a better approximation to f on crit(e) than R is. By Theorem 2.2, $R \notin B^{\ell}_{m,n}(f)$. ∎

THEOREM 3.4. *Let f be a real continuous function on* $[1, 1]$ *and let* $R = q/p \in \Pi^r_{m,n}$, $(q, p) = 1$ *be such that the longest alternation set of* $f - R$ *has length* L.

(a) *If* $L \geqslant 2 + n + \max\{m + \deg p, n + \deg q\}$ *then* $R \in B_{m,n}(f)$.

(b) *If* $L \leqslant m + n + 1$ *then* $R \notin B^{\ell}_{m,n}(f)$.

The constants in (a) *and* (b) *are the best possible in the following sense:*

Let $A_1(I_1, I_2, I_3, I_4)$ *and* $A_2(I_1, I_2, I_3, I_4)$ *be integer functions of four integer variables such that for any real continuous* f *and* $R = q/p \in \Pi^r_{m,n}$, $(q, p) = 1$ *with* L *the length of the longest alternation set of* $f - R$ *it follows that*

(c) $L \geqslant A_1(m, n, \deg q, \deg p)$ *implies that* $R \in B_{m,n}(f)$; *and*

(d) $L \leqslant A_2(m, n, \deg q, \deg p)$ *implies that* $R \notin B^{\ell}_{m,n}(f)$;

then

(e) $A_1(m, n, \deg q, \deg p) \geqslant 2 + n + \max\{m + \deg p, n + \deg q)\}$;

and

(f) $A_2(m, n, \deg q, \deg p) \leqslant m + n + 1$.

Proof. Condition (a) is (b) of Theorem 1.1. Condition (b) follows from Theorem 2.1. The remainder of the theorem follows from Corollary 2.4 and Theorem 3.3. ∎

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